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PROJECTIVE MODULES FOR $SL(2, 2^n)$

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1. Introduction

An important topic in block theory and the modular representations of finite groups is the algebra structure of modular group algebras. Since algebras can be described in terms of endomorphisms of projective modules, the structure of such modules is of great interest. Moreover, many of the cohomological problems of group theoretical interest can also be viewed as questions about projective modules and resolutions.

If G is a finite group of Lie type of characteristic p while F is an algebraically closed field of characteristic p then modules for the group algebra FG are of particular interest. In a previous paper [2][1] we showed how tensor products could be used to study projective modules for these types of algebras. In this paper we wish to carry out these ideas in a special case.

Suppose that $G = SL(2, p^n)$, the most rudimentary group of Lie type of characteristic p. The algebra FG is the direct sum of two indecomposable algebras if p is two, while it is the sum of three if p is odd. This gives a natural division into two cases, the first of which we shall study in this paper. Henceforth, $p = 2$, $G = SL(2, 2^n)$ and F is an algebraically closed field of characteristic two.

All FG-modules considered will be right FG-modules finite dimensional over F. We shall assume familiarity with the usual properties of direct sums, tensor products, induced modules and homomorphisms as used previously [2]. We shall also denote the number of elements in the set X by $|X|$.

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[1] Note that the Corollary of that paper is not true without added hypothesis. See the paper by Feit referred to there.

2. Results[2]

We shall begin by recalling the well-known facts about simple FG-modules. Since F is algebraically closed it contains a unique subfield F_0 with 2^n elements; we may as well assume that $G = SL(2, F_0)$. The space of row vectors of length two over F now becomes a module for FG via multiplication of a vector by a matrix. This is a simple FG-module which we shall denote by V_1. If $0 \leq i < n$ then the taking of 2^i-th powers is an automorphism of F_0. This allows us to construct other two dimensional simple FG-modules. If v is a row vector of length two and g is in G then multiplication of v and the matrix obtained from g by replacing each entry by its 2^i-th power gives us such a module which we shall denote by V_{i+1}. Notice that if we do the same construction for $i = n$ then we just get V_1 again.

Let $N = \{1, 2, \ldots, n\}$ and where necessary regard these numbers as the distinct elements of Z/nZ so that $i+1$ always makes sense if i is in N. For any subset I of N let

$$V_I = \bigotimes_{i \in I} V_i$$

with the usual convention that if I is the empty set, then $V_\emptyset$ is the trivial FG-module F. We shall write $V_0 = V_\emptyset$ and often use 0 in place of $\emptyset$. The 2^n modules V_I are all simple FG-modules, one from each of the 2^n isomorphism classes of such modules. Moreover, the Steinberg module V_N is also projective.

The elements of G can be simultaneously conjugated into the transpose of their inverses. Therefore, the module V_1 is self-dual and so are all the algebraic conjugates V_i, $i \in N$, of V_1 also self-dual. Thus, each tensor product of such modules is again self-dual. In particular, if V is any simple FG-module then $V \simeq V^*$.

For each subset I of N let P_I be a projective cover of V_I so that P_I is an indecomposable projective FG-module. There is exactly one simple quotient module of P_I and this is isomorphic with V_I. Since V_N is already projective, we have $P_N \simeq V_N$.

Our first result is implied by previous work [4, 5] but we shall derive it as we go along in a natural development. For any subset I of N we shall always denote the complement of I in N by $\bar{I}$.

Theorem 1. *If I is a proper subset of N, then*

$$V_I \otimes V_N \simeq P_{\bar{I}}$$

while

$$V_N \otimes V_N \simeq P_0 \oplus P_N.$$

For any two subsets I and J of N let C_{IJ} be the corresponding entry of the Cartan matrix C so that C_{IJ} is the multiplicity of V_I as composition factor of P_J. Our next result describes C giving us more information about projective modules.

[2] These results were described in [1].

Theorem 2. *If I and J are subsets of N, then*

$$C_{IJ} = 2^{n-|I\cup J|}$$

or

$$C_{IJ} = 0$$

according as I and J do or do not satisfy the following conditions:

(i) *Whenever k is in $I\cap J$ and $k+1$ is not then $k+1$ is in neither I nor J;*

(ii) *If one of I and J is N, then the other is not empty.*

This gives a combinatorial description of the entries of this important matrix. Our next result also involves a combinatorial formula. It describes some cohomology groups which also give information about the structure of projective FG-modules. In fact, let M_I be the unique maximal submodule of P_I so $P_I/M_I \simeq V_I$. The module M_I has a largest semi-simple quotient module. The multiplicity of V_J, for a subset J of N, as a composition factor of this quotient equals

$$\dim_F \mathrm{Ext}^1_{FG}(V_I, V_J).$$

This equality is immediate from the long exact sequence for Ext corresponding to the short exact sequence $0 \to M_I \to P_I \to V_I \to 0$.

Theorem 3. *If I and J are subsets of N, then* $\mathrm{Ext}^1_{FG}(V_I, V_J) = 0$, *unless* $|I\cap J|+1 = |I\cup J| < n$ *and whenever k is in $I\cup J$ and $k-1$ is not in $I\cap J$, then $k-1$ is in neither I nor J, in which case*

$$\mathrm{Ext}^1_{FG}(V_I, V_J) = F.$$

Our knowledge of projective FG-modules allows us to prove some things about the structure of the algebra FG.

Theorem 4. *If R is the radical of FG and $n>1$, then $R^{2n} \neq 0$ and $R^{2n+1} = 0$.*

If $n = 1$ then, in fact $R \neq 0$ and $R^2 = 0$.

Throughout this paper we shall be dealing with tensor products of simple modules. Our last result deals with these products in general. Recall [1] that an indecomposable FG-module is simply generated if it is isomorphic with a summand of a tensor product of simple FG-modules. We shall now define some more modules. If I and J are subsets of N with I a subset of J, then set $V_{I,J} = V_I \otimes V_J$ unless $I = J = N$, in which case set $V_{N,N} = P_0$. Thus, if $I = 0$, then $V_{0,J} \simeq V_j$ while if $J = N$, then $V_{I,N} \simeq P_{\bar{I}}$, by Theorem 1.

Theorem 5. *The above modules have the following properties:*

(i) *The FG-modules $V_{I,J}$ are simply generated;*

(ii) *They are pairwise non-isomorphic and any simply generated FG-module is isomorphic with one of them;*

(iii) *The subring of the Green ring spanned by the isomorphism classes of simply generated FG-modules is semi-simple.*

The Green ring is the ring which has a Z-basis consisting of the isomorphism classes of indecomposable FG-modules with multiplication coming from tensor products. It is easy to see that the simply generated modules span a subring.

3. Preliminary results

In this section we shall prove a number of lemmas that will be used in the proofs of the theorems. The first three deal with the Brauer characters of G. For each subset I of N let φ_I be Brauer character of V_I so, in particular, φ_I is complex valued and defined on the elements of odd order in G. These characters are a Z-basis of a ring given by multiplication of these functions. If I, J and K are subsets of N then the coefficient of φ_K in the expression of $\varphi_I\varphi_J$ in terms of these characters equals the multiplicity of V_K as a composition factor of $V_I \otimes V_J$.

Lemma 1. *If i is in N, then $\varphi_i^2 = \varphi_{i+1} + 2\varphi_0$.*

Proof. Let H and A be cyclic subgroups of orders $2^n - 1$ and $2^n + 1$, respectively, of G. Every element of odd order of G is conjugate to an element of one of these subgroups so it suffices to show that both sides of the desired equality agree on these two subgroups.

However, there is a linear character λ of H such that the restriction $\varphi_i \,|\, H = \lambda + \bar{\lambda}$; this is because V_i has dimension two and there is an element of G which inverts H. Similarly, there is a linear character μ of A such that $\varphi_i \,|\, A = \mu + \bar{\mu}$. Thus,

$$\varphi_i^2 \,|\, H = \lambda^2 + \bar{\lambda}^2 + 2\lambda_0, \qquad \varphi_i^2 \,|\, A = \mu^2 + \bar{\mu}^2 + 2\mu_0,$$

where λ_0 and μ_0 are the principal characters of H and A, respectively.

The underlying vector space for both V_i and V_{i+1} is the space of row vectors of length two. If g is in G then the action of g on V_i is given by multiplication by the matrix obtained from g by replacing each entry by its 2^{i-1}-th power. For V_{i+1} the appropriate power is the 2^i-th. Hence,

$$\varphi_{i+1} \,|\, H = \lambda^2 + \bar{\lambda}^2, \qquad \varphi_{i+1} \,|\, A = \mu^2 + \bar{\mu}^2$$

and the lemma is proved.

In the next two results we study the multiplicities which certain Brauer characters occur in expressing a product $\varphi_I\varphi_J$ in terms of the Brauer characters of the simple FG-modules. Note that φ_I, for a subset I of N, equals the product of all the φ_i for i in I.

Lemma 2. *If I and J are proper subsets of N, then $\varphi_I\varphi_J$ does not involve φ_N unless N is the disjoint union of I and J, in which case $\varphi_I\varphi_J = \varphi_N$.*

Proof. If I and J are disjoint, then $\varphi_I\varphi_J = \varphi_{I\cup J}$ and the result is clear. Thus, we may assume that I and J are not disjoint and we wish to prove that φ_N does not occur in the product $\varphi_I\varphi_J$.

Therefore, we may choose i in $I \cap J$. Let k in N be such that $i, i+1, \ldots, k$ are all in $I \cap J$ but $k+1$ is not. Without any loss of generality we may assume that $k+1$ is not in I. Let $K = \{i, \ldots, k\}$ so that

$$\varphi_I\varphi_J = \varphi_K^2\varphi_{I-K}\varphi_{J-K}.$$

But, by Lemma 1,

$$\varphi_K^2 = (\varphi_{i+1} + 2\varphi_0) \cdots (\varphi_{k+1} + 2\varphi_0)$$

so φ_K^2 is a sum of terms φ_L, where L is a subset of $\{i+1, \ldots, k+1\}$. Thus, $\varphi_I\varphi_J$ is a sum of terms $\varphi_L\varphi_{I-K}\varphi_{J-K}$. However, L and $I-K$ are disjoint since $k+1$ is not in I, so $\varphi_I\varphi_J$ is a sum of products of the form $\varphi_{(I-K)\cup L}\varphi_{J-K}$. Therefore, by an obvious induction on $|I|+|J|$ the lemma is established.

Lemma 3. *If I and J are subsets of N then the multiplicity of φ_0 in $\varphi_I\varphi_J$ is zero unless N is not the disjoint union of I and J and whenever k is not in $I \cup J$ but $k+1$ is then $k+1$ is in $I \cap J$, in which case the multiplicity is $2^{|I\cap J|}$.*

Proof. First, note that the hypotheses and conclusions depend only on $I \cup J$ and $I \cap J$ and that $\varphi_I\varphi_J = \varphi_{I\cup J}\varphi_{I\cap J}$. Hence, we may as well assume that I is a subset of J. In this case we want to show that the multiplicity of φ_0 in $\varphi_I\varphi_J$ is zero unless certain conditions hold, in which case the multiplicity should be $2^{|I|}$. These conditions are as follows: I is not empty or $J \neq N$; whenever k is not in J but $k+1$ is in J then $k+1$ is in I.

We shall now define a number of subsets of N. Let T_i, for each i in I, consist of i together with all k in N such that each of $i+1, \ldots, k$ lies in J and not in I. Thus, the sets T_i are mutually disjoint. Let R be the complement to their union in J so J is the disjoint union of all the T_i and R. For each i in I let S_i consist of all k in N such that $k-1$ is in T_i. The sets S_i are also mutually disjoint. Moreover, R and S_i are also disjoint. For if r is in R and S_i then $r-1$ would be in T_i. Since R is disjoint from I it would follow that r is also in T_i, a contradiction.

We can now also restate our aim. If $I = \emptyset$ and $J = N$, then the result is clear. Otherwise, we want to show the multiplicity of φ_0 in $\varphi_I\varphi_J$ is zero unless R is empty, in which case we want to see that the multiplicity is $2^{|I|}$. Indeed, if k is not in J but $k+1$ is in $J-1$ then $k+1$ is in R, being in none of the sets T_i. And if R is not empty we can choose k not in R with $k+1$ in R and then k is not in J – or else it would be in R – while $k+1$ is in J and not in I.

We now have

$$\varphi_I\varphi_J = \Big(\prod_{i\in I} \varphi_i\varphi_{T_i}\Big)\varphi_R.$$

Therefore, it suffices to show that each factor $\varphi_i\varphi_{T_i}$ is a sum of $2\varphi_0$ and terms φ_L where L is a non-empty subset of S_i. For then each term in the expansion of $\varphi_I\varphi_J$ will be a product of Brauer characters corresponding to disjoint subsets of N. However,

$$\varphi_i\varphi_{T_i} = \varphi_i\varphi_i\varphi_{i+1}\cdots\varphi_k$$

provided $T_i = \{i,\ldots,k\}$. Hence,

$$\begin{aligned}\varphi_i\varphi_{T_i} &= (2\varphi_0+\varphi_{i+1})\varphi_{i+1}\cdots\varphi_k\\ &= 2\varphi_{i+1}\cdots\varphi_k + 2\varphi_{i+1}\varphi_{i+1}\cdots\varphi_k.\end{aligned}$$

However, the first term is of the desired sort and the second is one to which this argument can be repeated. The desired sum results.

Lemma 4. *If $i\in N$, then $V_i\otimes V_i$ is a uniserial module with composition factors V_0, V_{i+1}, V_0 and in that order.*

(That is, $V_i\otimes V_i$ has a unique composition series and if it is $V_i\otimes V_i = U_1 \supset U_2 \supset U_3 \supset U_4 = 0$, then $U_1/U_2 \simeq U_3/U_4 \simeq V_0$ and $U_2/U_3 \simeq V_{i+1}$).

Proof. First, we observe that the composition factors are as stated as a consequence of Lemma 1. Moreover,

$$\begin{aligned}\mathrm{Hom}_{FG}(V_0, V_i\otimes V_i) &\simeq \mathrm{Hom}_{FG}(V_0\otimes V_i^*, V_i)\\ &\simeq \mathrm{Hom}_{FG}(V_i, V_i) = F\end{aligned}$$

so that $V_i\otimes V_i$ has a unique submodule S isomorphic with V_0. Since this tensor product is self-dual it has a unique maximal submodule M with quotient isomorphic with V_0. Moreover,

$$\begin{aligned}\mathrm{Hom}_{FG}(V_{i+1}, V_i\otimes V_i) &\simeq \mathrm{Hom}_{FG}(V_{i+1}\otimes V_i^*, V_i)\\ &\simeq \mathrm{Hom}_{FG}(V_{\{i,i+1\}}, V_i) = 0\end{aligned}$$

so that $V_i\otimes V_i$ has no submodule isomorphic with V_{i+1}. By duality there is no quotient module isomorphic with V_{i+1}. Thus, S is the only simple submodule of $V_i\otimes V_i$ and M is the only maximal submodule. Therefore, M must contain S and since $V_i\otimes V_i$ has only three composition factors it follows that $V_i\otimes V_i \supset M \supset S \supset 0$ is the only composition series and the factors are as claimed.

Lemma 5. *If $i\in N$, then $V_i\otimes V_i\otimes V_i \simeq V_i\oplus V_i\oplus V_{\{i,i+1\}}$.*

Proof. First, note that

$$\begin{aligned}\varphi_i^3 &= \varphi_i^2\varphi_i = (2\varphi_0 + \varphi_{i+1})\varphi_i \\ &= 2\varphi_i + \varphi_{\{i,i+1\}}\end{aligned}$$

and so the composition factors of the triple tensor product are as stated. Also

$$\begin{aligned}\mathrm{Hom}_{FG}(V_i, V_i \otimes V_i \otimes V_i) &\simeq \mathrm{Hom}_{FG}(V_i \otimes V_i^*, V_i \otimes V_i) \\ &\simeq \mathrm{Hom}_{FG}(V_i \otimes V_i, V_i \otimes V_i).\end{aligned}$$

However, this last vector space has a basis consisting of the identity endomorphism and an endomorphism which, in the notation of the proof of the previous lemma, maps $V_i \otimes V_i$ onto S with kernel M. Hence, $V_i \otimes V_i \otimes V_i$ has a submodule X isomorphic with $V_i \oplus V_i$. By duality it has a submodule Y with quotient also isomorphic with $V_i \oplus V_i$. Since V_i is a composition factor with multiplicity two it follows that $X \cap Y = 0$ and that $V_i \otimes V_i \otimes V_i$ is the direct sum of X and Y. Again, because we know the composition factors, we must have Y isomorphic with $V_{\{i,i+1\}}$ and the result holds.

We shall examine the submodules of one more type of module, the modules $V_{j,J}$ defined in section two for $j \in J$ and J a subset of N. Thus, $V_{j,J} = V_j \otimes V_J$. We begin by defining some subsets of J and then an associated sequence of subsets of N. Choose an integer k with $0 \leqslant k < n$ maximal such that $j+i$ is in J for every i with $0 \leqslant i \leqslant k$. Thus, $j, j+1, \ldots, j+k$ are in J and either $k = n-1$ and $J = N$ or $j+k+1$ is not in J. For each i with $0 \leqslant i \leqslant k$, let T_i be the set $\{j+i, \ldots, j+k\}$ so $T_0 \supset T_1 \supset \cdots \supset T_k$. Set $R = J - T_0$. The sequence of subsets of N which we wish to associate with the pair j, J is defined now as follows:

$$T_1 \cup R, T_2 \cup R, \ldots, T_k \cup R, R, \{k+1\} \cup R, R, T_k \cup R, \ldots, T_2 \cup R, T_1 \cup R.$$

We can now describe $V_{j,J}$.

Lemma 6. *If j is an element of the subset J of N then the module $V_{j,J}$ is uniserial. Moreover, the composition factors in the composition series of $V_{j,J}$ are, in order, the simple modules corresponding to the subsets of N in the sequence of subsets of N associated to the pair j, J.*

Proof. We proceed by induction on k. If $k = 0$, then $J = \{j\} \cup R$ and $j+1$ is not in J. Moreover,

$$V_{j,J} = V_j \otimes V_{\{j\} \cup R} \simeq (V_j \otimes V_j) \otimes V_R$$

so that, by Lemma 4, $V_{j,J}$ has a series of submodules with successive quotients being $V_0 \otimes V_R$, $V_{\{j+1\}} \otimes V_R$, $V_0 \otimes V_R$, that is V_R, $V_{\{j+1\} \cup R}$, V_R. In particular, this series is a composition series of $V_{j,J}$; the lemma will be proved once we show this series of submodules is the only composition series of $V_{j,J}$. Since this module is self dual we need only show that there is no submodule isomorphic with $V_{\{j+1\}\cup R}$. However,

$$\begin{aligned}\mathrm{Hom}_{FG}(V_{\{j+1\}\cup R}, V_{j,J}) &\simeq \mathrm{Hom}_{FG}(V_{\{j+1\}\cup R}\otimes V_j^*, V_J)\\ &\simeq \mathrm{Hom}_{FG}(V_{\{j,j+1\}\cup R}, V_J)=0\end{aligned}$$

so this case is established.

Now suppose that $k\neq 0$. We have that

$$\begin{aligned}V_{j,J} &\simeq V_j\otimes V_{T_0}\otimes V_R\\ &\simeq (V_j\otimes V_j)\otimes V_{T_1}\otimes V_R \simeq (V_j\otimes V_j)\otimes V_{T_1\cup R}.\end{aligned}$$

Therefore, by Lemma 4, $V_{j,J}$ has a series of submodules

$$V_{j,J}\supset M\supset S\supset 0$$

with successive quotients isomorphic with $V_{T_1\cup R}$, $V_{j+1}\otimes V_{T_1\cup R}$, $V_{T_1\cup R}$. However, the middle term is just $V_{j+1,J-\{j\}}$ and induction applies to it. The sequence of subsets of N associated to the pair $j+1, J-\{j\}$ is easily seen to be as follows:

$$T_2\cup R,\ldots, T_k\cup R, R, \{k+1\}\cup R, R, T_k\cup R,\ldots, T_2\cup R.$$

Hence, M/S is uniserial with the composition factors in its unique composition series being the simple modules corresponding to the subsets in this sequence. Hence, it suffices to show that S is the only simple submodule and M is the only maximal submodule of $V_{j,J}$. By duality it is enough to deal with S. Let T be another simple submodule of $V_{j,J}$. From the structure of the module it must be isomorphic with $V_{T_2\cup R}$ or $V_{T_1\cup R}$. However,

$$\begin{aligned}\mathrm{Hom}_{FG}(V_{T_2\cup R}, V_{j,J}) &\simeq \mathrm{Hom}_{FG}(V_{\{j\}\cup T_2\cup R}, V_J)\\ &\simeq \mathrm{Hom}_{FG}(V_{J-\{j+1\}}, V_J)=0.\end{aligned}$$

Therefore, we must have T isomorphic with S. It follows that M/S is isomorphic with a quotient of $V_{j,J}$, namely $V_{j,J}/S+T$, and so $V_{T_2\cup R}$ is a homomorphic image of $V_{j,J}$. By duality this is a contradiction.

The last result of this section is only to be used in giving a second proof of Theorem 1.

Lemma 7. *If A is a cyclic subgroup of order 2^n+1 in G then the FG-module induced by the trivial FA-module F is isomorphic with P_0.*

Proof. Since A has odd order every FA-module is certainly projective. Thus, the induced module F^G is a projective FG-module. It is thus a sum of indecomposable projective FG-modules. Therefore, it suffices to show that $\mathrm{Hom}_{FG}(F^G, V_I)$ is zero for every nonempty subset of N but is one domensional for $I=\emptyset$. But

$$\mathrm{Hom}_{FG}(F^G, V_I)\simeq \mathrm{Hom}_{FA}(F, V_I\,|\,A).$$

Since $V_0|A \simeq F$ it suffices to show that if I is not empty then $\varphi_I|A$ does not involve the principal character of A.

As in Lemma 1, $\varphi_1|A = \mu + \bar{\mu}$ where μ is a faithful linear character of A. Hence,

$$\varphi_I|A = \prod_{i \in I} (\mu^{2^{i-1}} + \bar{\mu}^{2^{i-1}}).$$

But μ has multiplicative order $2^n + 1$ so no term in the expansion of this product is the principal character of A.

4. Proofs

In this section we shall supply the proofs of the five theorems. We shall begin with the first theorem and give two different proofs for it.

First, observe that if J is a subset of N then

$$\begin{aligned}\mathrm{Hom}_{FG}(V_J \otimes V_N, V_{\bar{J}}) &\simeq \mathrm{Hom}_{FG}(V_N, V_J^* \otimes V_{\bar{J}})\\ &\simeq \mathrm{Hom}_{FG}(V_N, V_N) = F\end{aligned}$$

so that $P_{\bar{J}}$ is a direct summand of $V_J \otimes V_N$. In particular, $\dim_F P_{\bar{J}} \leq 2^{n+|J|}$. If we can establish equality here then we will also have that $P_{\bar{J}} \simeq V_N \otimes V_J$.

When $J = N$ we can make an improvement. We shall see that P_N as well as P_0 is isomorphic with a direct summand of $V_N \otimes V_N$. Indeed,

$$\begin{aligned}\varphi_N\varphi_N &= \varphi_1^2\varphi_2^2 \cdots \varphi_n^2 = (2\varphi_0 + \varphi_2) \cdots (2\varphi_0 + \varphi_1)\\ &= \varphi_1\varphi_2 \cdots \varphi_n + \cdots = \varphi_N + \cdots\end{aligned}$$

so that V_N is a composition factor of $V_N \otimes V_N$. However, V_N is projective and therefore it is injective and must be a direct summand of $V_N \otimes V_N$. This yields the inequality $\dim_F P_0 \leq 2^{2n} - 2^n$. If we show that this too is an equality, then we will have that $V_N \otimes V_N \simeq P_0 \oplus V_N$. This will establish Theorem 1.

However, using the decomposition of the free FG-module FG into indecomposable FG-modules and the above estimates for dimensions we have

$$\begin{aligned}|G| &\leq \sum_{I \subseteq N} (\dim_F V_I)(\dim_F P_I)\\ &= \Big(\sum_{I \subseteq N} (2^{|I|})(2^{n+(n-|I|)})\Big) - 2^n = 2^n \cdot 2^{2n} - 2^n = |G|.\end{aligned}$$

This forces all the inequalities to be equalities and the theorem is proved.

Now we shall give another proof of this same theorem. Just as in the other proof we know that P_0 and P_N are both summands of $V_N \otimes V_N$. However, from Lemma 7 we have

$$\dim_F P_0 = |G : A| = 2^{2n} - 2^n$$

so $\dim_F P_0 + \dim_F P_N = \dim_F(V_N \otimes V_N)$ and we have the required decomposition of $V_N \otimes V_N$.

We must now deal with $V_J \otimes V_N$ where J is a proper subset of N. Part of the above proof shows that in a decomposition of $V_J \otimes V_N$ into indecomposable modules the module $P_{\bar{J}}$ occurs just once. Suppose that I is a subset of N other than $\bar{J}$; we want to see that P_I is not a summand of $V_J \otimes V_N$. That is, we desire that $\mathrm{Hom}_{FG}(V_J \otimes V_N, V_I) = 0$.

But, if $I = N$, then J is not empty and

$$\mathrm{Hom}_{FG}(V_J \otimes V_N, V_N) \simeq \mathrm{Hom}_{FG}(V_J, V_N \otimes V_N).$$

The scale of $V_N \otimes V_N$ is isomorphic with $V_0 \oplus V_N$ so this space of maps is zero, as desired.

If I is a proper subset of N, then

$$\mathrm{Hom}_{FG}(V_J \otimes V_N, V_I) \simeq \mathrm{Hom}_{FG}(V_N, V_J \otimes V_I).$$

But V_N is not even a composition factor of $V_J \otimes V_I$, by Lemma 2, so the theorem is proved again.

We now turn to Theorem 2 and we have to calculate $\dim_F \mathrm{Hom}_{FG}(P_I, P_J)$ for any two subsets I and J of N. We shall do this in three steps.

First, suppose that I and J are both empty. Since $V_N \otimes V_N \simeq P_0 \oplus P_N$ it follows that $C_{0,0}$ is the multiplicity of V_0 as a composition factor of $V_N \otimes V_N$. However, Lemma 3 gives this multiplicity as 2^n and the theorem is proved in this case.

Next, suppose that I is empty and J is not. If $J = N$, then $\mathrm{Hom}_{FG}(P_I, P_J) = 0$ and the theorem holds; hence, we may assume that J is a proper non-empty subset of N. Thus,

$$\mathrm{Hom}_{FG}(P_0, P_J) \simeq \mathrm{Hom}_{FG}(P_0, V_{\bar{J}} \otimes V_N)$$

which has dimension the multiplicity of V_0 as a composition factor of $V_{\bar{J}} \otimes V_N$. This is just $2^{|\bar{J}|}$ by Lemma 3 which is $2^{n-|I \cup J|}$ as desired.

Since the Cartan matrix is symmetric we can now assume that both I and J are non-empty. Thus,

$$\begin{aligned} \mathrm{Hom}_{FG}(P_I, P_J) &\simeq \mathrm{Hom}_{FG}(V_{\bar{I}} \otimes V_N, V_{\bar{J}} \otimes V_N) \\ &\simeq \mathrm{Hom}_{FG}(V_N \otimes V_N, V_{\bar{I}} \otimes V_{\bar{J}}) \\ &\simeq \mathrm{Hom}_{FG}(P_0 \oplus V_N, V_{\bar{I}} \otimes V_{\bar{J}}). \end{aligned}$$

If N is the disjoint union of I and J then $V_{\bar{I}} \otimes V_{\bar{J}} \simeq V_N$ and the last space of homomorphisms is one dimensional, as desired. Otherwise, by Lemma 2, V_N is not a composition factor of $V_{\bar{I}} \otimes V_{\bar{J}}$ so

$$\mathrm{Hom}_{FG}(P_I, P_J) \simeq \mathrm{Hom}_{FG}(P_0, V_{\bar{I}} \otimes V_{\bar{J}})$$

which has dimension the multiplicity of V_0 as composition factor of $V_{\bar{I}} \otimes V_{\bar{J}}$.

By Lemma 3, this multiplicity is zero, unless whenever k is not in $\bar{I} \cup \bar{J}$ but $k+1$ is then $k+1$ is in $\bar{I} \cap \bar{J}$, in which case the multiplicity is

$$2^{|\bar{I} \cap \bar{J}|} = 2^{n-|I \cup J|}.$$

This condition on k is that whenever k is in $I \cap J$ and $k+1$ is not in $I \cap J$ then $k+1$ is in neither I nor J, just as required.

We turn our attention to the third theorem. First, note that the hypotheses and conclusions depend only on $I \cup J$ and $I \cap J$. Moreover,

$$\begin{aligned} \mathrm{Ext}^1_{FG}(V_I, V_J) &\simeq \mathrm{Ext}^1_{FG}(V_I, V_{I \cap J} \otimes V_{J-(I \cap J)}) \\ &\simeq \mathrm{Ext}^1_{FG}(V_I \otimes V_{J-(I \cap J)}, V_{I \cap J}) \\ &\simeq \mathrm{Ext}^1_{FG}(V_{I \cup J}, V_{I \cap J}). \end{aligned}$$

Hence, replacing I and J by $I \cup J$ and $I \cap J$ respectively, we may assume without any loss of generality that I contains J. If $I = N$, then the cohomology group in question vanishes as V_N is projective and the result holds; hence, we may also assume that I is a proper subset of N.

Suppose that I and J do satisfy the conditions described in the theorem. This means that I consists of J together with one more element k of N and that $k-1$ is not in J (and thus certainly not in I). Let us see that $\mathrm{Ext}^1_{FG}(V_I, V_J) \neq 0$. Consider the module $V_{k-1,\{k-1\} \cup I}$. Lemma 6 yields that this module has a quotient module which is a non-split extension of the simple module V_I by the submodule $V_{I-\{k\}} = V_J$. The desired non-vanishing holds.

If $|I| = n-1$, then $I = N - \{r\}$ for some r in N. By Theorem 1 we have $P_I \simeq V_r \otimes V_N = V_{r,N}$. However, the structure of this module is given by Lemma 6. In fact, the largest semi-simple quotient of the maximal submodule M_I of $P_I \simeq V_{r,N}$ is isomorphic with $V_{I-\{r+1\}}$. Hence,

$$\mathrm{Ext}^1_{FG}(V_I, V_{I-\{r+1\}}) = F$$

while if J is a subset of N other than $I - \{r+1\}$, then $\mathrm{Ext}^1_{FG}(V_I, V_J) = 0$. This is exactly what is required in Theorem 3 in this case. Hence, we may now assume that $|I| < n-1$.

Therefore, to complete the proof of this theorem, it is sufficient to show that there is s in N which is not in I such that

$$\dim_F \mathrm{Ext}^1_{FG}(V_I, V_J) \leq \dim_F \mathrm{Ext}^1_{FG}(V_{I \cup \{s\}}, V_{J \cup \{s\}})$$

and that I and J satisfy the conditions given in the theorem if, and only if, $I \cup \{s\}$ and $J \cup \{s\}$ satisfy them. Indeed, once we show all this the theorem will follow from an immediate downward induction together with the non-vanishing we have already proved.

Suppose that $\dim_F \mathrm{Ext}^1_{FG}(V_I, V_J) = d$ so that there is a module X which has a unique maximal submodule Y such that $X/Y \simeq V_I$ and Y is isomorphic with a direct sum of d copies of V_J. If s is an element of N not in I, $X' = X \otimes V_s$, $Y' = Y \otimes V_s$,

$I' = I \cup \{s\}$ and $J' = J \cup \{s\}$ then $X'/Y' \simeq V_{I'}$ and Y' is isomorphic with the direct sum of d copies of $V_{J'}$. If, for suitable s, Y' is the unique maximal submodule of X' then we will have

$$\dim_F \operatorname{Ext}^1_{FG}(V_{I'}, V_{J'}) \geq d = \dim_F \operatorname{Ext}^1_{FG}(V_I, V_J).$$

To show that Y' has this property it suffices to prove that $\dim_F \operatorname{Hom}_{FG}(X', V_{J'})$ is one if $I = J$ and zero otherwise.

In any case,

$$\operatorname{Hom}_{FG}(X', V_{J'}) \simeq \operatorname{Hom}_{FG}(X \otimes V_s, V_J \otimes V_s)$$
$$\simeq \operatorname{Hom}_{FG}(X, V_s \otimes V_s \otimes V_J) \simeq \operatorname{Hom}_{FG}(X, V_{s,J'}).$$

We shall now consider the cases $I = J$, $|I| > |J| + 1$ and $|I| = |J| + 1$ in turn showing we can choose s suitably.

If $I = J$ then I and J do not satisfy the conditions of the theorem and neither do I' and J' as $I' = J'$. Moreover, by Lemma 6, the only submodule of $V_{s,I'} = V_{s,J'}$ having every composition factor isomorphic with $V_I = V_J$ is the socle so $\operatorname{Hom}_{FG}(X, V_{s,J'})$ is one dimensional. Thus, in this case, any element of $N - I$ will do for s.

Next, suppose that $|I| > |J| + 1$. Again I and J as well as I' and J' do not satisfy the conditions of the theorem. Furthermore, by Lemma 6, every composition factor of $V_{s,J'}$ is of the form V_K for a subset K of N with $|K| \leq |J'| < |I|$ so there are no non-zero homomorphisms from X to $V_{s,J'}$. Again, s can be any element of $N - I$.

Finally, suppose that $|I| = |J| + 1$ so that $I = J \cup \{k\}$ for some k in N. Thus, since the socle of $V_{s,J'}$ is just V_J the only way there can be a non-zero homomorphism of X to $V_{s,J'}$ is that the composition factor of $V_{s,J'}$ just above V_J is V_I. But $|I| > |J|$ so this will happen exactly when $s+1$ is not in J, $V_{s,J'}$ has exactly three composition factors and they are, in order, V_J, $V_{\{s+1\} \cup J}$, V_J. Thus, we want that $s+1$ is not k and that s is also chosen so that I and J satisfy the conditions of the theorem if, and only if, I' and J' do. If I and J do not satisfy the conditions, that is, $k-1$ is in J then simply choose s as any element of $N - I$. Thus, $s + 1 \neq k$ and I' and J' do not satisfy the conditions as $I' = J' \cup \{k\}$ and $k - 1$ is in J'. If I and J do satisfy the conditions then $k-1$ is not in J. Now choose s to be any element of $N - I$ other than $k-1$; we may do this as $|I| < n - 1$. It follows that $s + 1 \neq k$ and that I' and J' do satisfy the conditions since $I' = J' \cup \{k\}$ with $k-1$ not in J'. The third theorem is now proved.

If U is an FH-module for a group H then the lower Loewy series of U is the sequence of submodules

$$U, RU, R^2U, \cdots$$

where R is the radical of FH. Thus, RU is the smallest submodule of U with semi-simple quotient, R^2U is the smallest submodule of RU such that the quotient of RU by it is semi-simple, and so on. The least integer k such that $R^kU = 0$ is the Loewy length of U. Setting $R^0U = U$ we let $L_i(U) = R^i(U)/R^{i+1}(U)$, $i \geq 0$, so that U has Loewy length k if, and only if, $L_{k-1}(U) \neq 0$ and $L_k(U) = 0$. That is, if, and only if, there are exactly k non-zero factors $L_i(U)$.

If $U_1, \ldots, U_s$ are modules for the group algebras $FH_1, \ldots, FH_s$, respectively, and we set $H = H_1 \times H_2 \times \cdots \times H_s$, $U = U_1 \otimes \cdots \otimes U_s$ then U is an FH-module and [3]

$$L_t(U) \simeq \bigoplus (L_{t_1}(U) \otimes \cdots \otimes L_{t_s}(U))$$

where the sum is over all s-tuples of non-negative integers $t_1, \ldots, t_s$ whose sum is t. In particular, the Loewy length of U is the sum of the Loewy lengths of the U_i less $s-1$. Suppose now that all the groups H_i are equal, being the group G_0. Moreover, suppose that the tensor product of composition factors, one from each U_i, is again a simple FG_0-module. Thus, if we regard U as an FG_0-module, via the usual diagonal action, then each factor $L_t(U)$ is a semi-simple FG_0-module. Thus, the FG_0-module U has a series of submodules, with semi-simple factors, the number of these being the sum of the Loewy lengths of the U_i less $s-1$. Therefore, the Loewy length of U is at most the sum of the Loewy lengths of the U_i less $s-1$.

In order to prove Theorem 4, we must show that there is an indecomposable projective FG-module of Loewy length $2n+1$ and that no such module has Loewy length exceeding $2n+1$. However, $P_n \simeq V_n \otimes V_N = V_{n,N}$ is a uniserial module with $2n+1$ composition factors, by Lemma 6. Since every indecomposable projective FG-module is a direct summand of some module $V_I \otimes V_N$ for a non-empty subset I of N, we need only show that each such tensor product has a Loewy length at most $2n+1$. For each i in I let T_i be the subset of N consisting of i and all elements j of N between i and the next element of i, where "next" means in the usual circular ordering of N. Let S_i consist of all j in N such that $j-1$ is in T_i. Thus,

$$V_I \otimes V_N \simeq \bigotimes_{i \in I} V_{i,T_i}.$$

Moreover, V_{i,T_i} is of Loewy length $2|T_i|+1$ and all its composition factors are of the form V_K for a subset K of S_i, by Lemma 6. However all the sets S_i are mutually disjoint so the tensor product of composition factors, one from each V_{i,T_i}, is again a simple FG-module. By our above remarks, $V_I \otimes V_N$ has Loewy length bounded by

$$\left(\sum_{i \in I} (2|T_i|+1)\right) - (|I|-1)$$

which is just $2n+1$.

The last task is to prove Theorem 5. Let's begin by showing that each of the FG-modules $V_{I,J}$ is indecomposable and thus is certainly simply generated. This holds if $J = N$ by Theorem 1. If J is a proper subset of N, then

$$V_{I,J} \otimes V_{\bar{J}} \simeq V_I \otimes V_J \otimes V_{\bar{J}} \simeq V_I \otimes V_N \simeq V_{I,N}$$

since I is a proper subset of N as it is contained in J. Hence, $V_{I,J}$ is certainly indecomposable as the tensor product of it and another module is indecomposable.

We next shall show that $V_{I,J}$ has a simple socle and this is isomorphic with $\check{V}_{J-I}$. If $J = N$ this is correct as $V_{I,N}$ is the projective cover of $V_{\bar{I}}$. If J is a proper subset of N, then the tensor product of $V_{I,J}$ and $V_{\bar{J}}$ has a simple socle, by the above calculation, so that so does $V_{I,J}$. Moreover,

$$\begin{aligned}\mathrm{Hom}_{FG}(V_{J-I}, V_{I,J}) &\simeq \mathrm{Hom}_{FG}(V_{J-I}, V_I \otimes V_J)\\ &\simeq \mathrm{Hom}_{FG}(V_{J-I} \otimes V_I, V_J)\\ &\simeq \mathrm{Hom}_{FG}(V_J, V_J) = F\end{aligned}$$

so $V_{I,J}$ does have a submodule isomorphic with V_{J-I} and our claim holds.

We also claim that no two of the modules $V_{I,J}$ are isomorphic. In fact, suppose that $I' \subseteq J'$ and that $V_{I,J} \simeq V_{I',J'}$; we must show that $I = I'$ and $J = J'$. The previous paragraph shows that certainly $I - J = I' - J'$; call this subset K. It remains to show that $I = I'$. Without any loss of generality, we may assume that there is i in I not in I'; we need only derive a contradiction. It follows that i is not in $J - I = K$ so it is not in $I' \cup K = J'$. Now

$$V_{I,J} \otimes V_i \simeq (V_i \otimes V_i \otimes V_i) \otimes V_{I-\{i\}} \otimes V_{I-\{i\}} \otimes V_K$$

is not indecomposable as the first factor is the direct sum of three modules, by Lemma 5. On the other hand

$$\begin{aligned}V_{I',J'} \otimes V_i &\simeq V_{I'} \otimes V_{I'} \otimes V_K \otimes V_i\\ &\simeq V_{I'} \otimes V_{I'} \otimes V_{K \cup \{i\}} \simeq V_{I',J' \cup \{i\}}\end{aligned}$$

(as $J' \cup \{i\} = N$ implies $I' \neq N$) which is indecomposable, a contradiction to the supposed isomorphism.

Let us now show that every simply generated module is isomorphic with one of the modules $V_{I,J}$. Since these modules are simply generated and since every simple module is a tensor product of two dimensional modules, it suffices to show that $V_{I,J} \otimes V_k$ is always a direct sum of modules of the right sort whenever $k \in N$ and $I \subseteq J \subseteq N$.

If k is not in J then $V_{I,J} \otimes V_k \simeq V_{I,J \cup \{k\}}$ (as I is certainly a proper subset of N in this case). If k is in J but not in I then $V_{I,J} \otimes V_k \simeq V_{I \cup \{k\},J}$ so that leaves only the case of $k = i \in I$ to deal with. But then

$$\begin{aligned}V_{I,J} \otimes V_i &\simeq (V_i \otimes V_i \otimes V_i) \otimes V_{I-\{i\}} \otimes V_{J-\{i\}}\\ &\simeq (V_i \oplus V_i \oplus V_{\{i,i+1\}}) \otimes V_{I-\{i\}} \otimes V_{J-\{i\}}\\ &\simeq V_{I-\{i\},J} \oplus V_{I-\{i\},J} \oplus (V_{I-\{i\},J} \otimes V_{i+1}).\end{aligned}$$

The first two terms are of the desired sort and the third is of the type we are now studying. But since $|I| + |J| > |I - \{i\}| + |J|$ we are done by an obvious induction.

We now turn to the semi-simplicity statement of the theorem. Let R be the commutative ring with unit element generated by elements $x_1, \ldots, x_n$ and defining relations $x_i^3 = 2x_i + x_i x_{i+1}$, for all $i \in N$. It follows that R is spanned by all the elements

$$x_1^{a_1} \cdots x_n^{a_n}$$

where each a_i is zero, one or two. If we set $x_I = \prod_{i \in I} x_i$ for any subset I of N then this spanning set consists of the elements $x_I x_J$ for all pairs of subsets I and J of N with $I \subseteq J$. The empty product $x_\emptyset$ is, of course, taken to be the unit element 1 of R.

Let S be the [illegible] spanned by the isomorphism classes of simply generated modules so it is spanned by the isomorphism classes of the modules $V_{I,J}$. Thus, it has a Z-basis consisting of the isomorphism classes of the modules $V_I \otimes V_J$. Hence, S is generated by the isomorphism classes of the modules V_i, $i \in N$. Because of Lemma 5, there is a homomorphism of R onto S sending each x_i to the isomorphism class of V_i. The spanning set of R and the basis of S we have constructed are parametized by the pairs I, J of subsets of N with $I\,J$. Hence, this homomorphism is an isomorphism of R onto S. Thus, we need only prove that R is semi-simple.

Because of the homomorphism we also have that R has a Z-basis consisting of all the elements $x_I x_J$, $I \subseteq J \subseteq N$. Therefore, to demonstrate the semi-simplicity of R we shall construct distinct homomorphisms of R to the complex numbers parametized by these same pairs I, J. To do this we need only construct, for each $I \subseteq N$, $2^{n-|I|}$ distinct homomorphisms such that homomorphisms belonging to different subsets I never coincide. We shall do this by constructing for each I exactly $2^{n-|I|}$ homomorphisms which vanish on x_k if, and only if, k is in I.

Constructing homomorphisms is exactly the same as finding "solutions" of the defining relations for R, that is, finding n complex numbers z_i, $i \in N$, such that $z_i^3 = 2z_i + z_i z_{i+1}$ for all i. For every subset I of N we shall find exactly $2^{n-|I|}$ such solutions in which $z_k = 0$ if, and only if, k is in I.

First, we shall deal with $I = \emptyset$. If g is any element of odd order in G, then $\varphi_j(g) \neq 0$ for any $j \in N$ as the proof of Lemma 1 shows. The numbers $z_j = \varphi_j(g)$ are then a solution, by Lemma 1. If g and h are non-conjugate elements of odd order, then the corresponding solutions can not coincide; for this would imply that the table of Brauer characters had a zero determinant. Since there are 2^n conjugacy classes of elements of odd order in G the case $I = \emptyset$ is dealt with.

Next, suppose that I is a non-empty proper subset of N. For each i in I let S_i be the set of elements of $\bar{I}$ between i and the next element of I so that the sets S_i form a partition of $\bar{I}$. Let's see what are the equations that the z_k must satisfy for k in one such segment S_i. Suppose that the next element of I after i is j so $S_i = \{i+1, \ldots, j-1\}$. These equations are

$$
\begin{aligned}
z_i^3 &= 2z_i + z_i z_{i+1}, \\
z_{i+1}^3 &= 2z_{i+1} + z_{i+1} z_{i+2}, \\
&\cdots \\
z_{j-1}^3 &= 2z_{j-1} + z_{j-1} z_j, \\
z_j^3 &= 2z_j + z_j z_{j+1}.
\end{aligned}
$$

Since we want $z_i = z_j = 0$ and all the $z_k \neq 0$ for $k \in S_i$ these equations reduce to the non-vanishing condition and

$$\begin{aligned} z_{j-1}^2 &= 2, \\ z_{j-2}^2 &= 2 + z_{j-1}, \\ &\dots \\ z_{i+1}^2 &= 2 + z_{i+2}. \end{aligned}$$

We can solve these: $z_{j-1} = \pm\sqrt{2}$, $z_{j-2} = \pm\sqrt{2 + z_{j-1}}, \dots, z_{i+1} = \pm\sqrt{2 + z_{i+2}}$. There are exactly $2^{|S_i|}$ solutions all non-zero: this is because if r is a real number strictly between zero and two then so are $\sqrt{2 \pm r}$. Since we can solve these equations for each S_i independently we thus have $2^{n-|I|}$ solutions with the z_k zero exactly when k is in I.

To conclude the proof we need only note that the solution with all the $z_k = 0$ is the one solution needed to deal with $I = N$ and the proof of the theorem is complete.

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